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1986 J. Phys. A: Math. Gen. 19 859

(http://iopscience.iop.org/0305-4470/19/6/016)

J. Phys. A: Math. Gen. 19 (1986) 859-863. Printed in Great Britain

The short-range behaviour of the deuteron wavefunctions with a regularised tensor potential

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Received 16 August 1985, in final form 7 October 1985

Abstract. It is shown that deuteron wavefunctions which arise from a regularised tensor potential, $V_T(r) \to$ constant as $r \to 0$, have a rather unusual short-range behaviour. The reason for the node in a recent model for the deuteron is made clear. A second regular solution is also found.

1. Introduction

Kukulin *et al* (1984) have presented a model for the nucleon-nucleon potential which has a forbidden state suggested from a six-quark model with one-pion exchange. In this model, the deuteron wavefunctions have a node at short distances. The latter is stated to be a consequence of the two-channel potential model with local potentials (Kukulin *et al* 1984). In this paper, we shall show that although this is the case, the D-state wavefunction at short distances behaves like (constant)$r^3 \ln r$ rather than (constant)r^2 from Reid-type potentials (with no small soft core) (Reid 1968) or (constant)r^3 for the parametrised Paris wavefunction (Lacombe *et al* 1981) and some non-local potentials (Friar and Fallieros 1984). This is because the tensor potential is regularised at the origin, i.e. $V_T(r) \to$ constant as $r \to 0$. However, the relative values of the constant are such that *w/u remains positive, regardless of the negative behaviour of $\ln r$.* We shall show also that, in principle, it is possible to have the behaviour $u \sim r^5$, $w \sim r^3$, but this would require a very special potential.

2. Deuteron wavefunctions

We shall not be concerned about the presence of the node in the wavefunction. The behaviour of $w(r)$ mentioned above holds for any model with a regularised tensor potential. The main emphasis in this paper is on the behaviour of the wavefunctions at short distances in the context of the Schrödinger equation and a potential model. To illustrate the procedure most clearly, we shall follow Kukulin *et al* in regularising the central potential also. Our results are also valid under the usual assumption that $V_C(r)$ is no more singular than r^{-1}. Thus, we can define

$$a_{11} = -\lim_{r\to 0} [(2m/\hbar^2) V_C(r) + \alpha^2]$$

$$a_{12} = -\lim_{r\to 0} [(2m/\hbar^2)\sqrt{8}\, V_T(r)]$$

$$a_{22} = -\lim_{r\to 0} [(2m/\hbar^2)(V_C(r) - 2V_T(r) - 3V_{LS}(r)) + \alpha^2]$$

where $-\hbar^2\alpha^2/2m$ is the binding energy of the deuteron.

For the potential of Kukulin *et al*

$$a_{ij} \simeq a_{ij}^{(0)} + a_{ij}^{(1)} r + a_{ij}^{(2)} r^2 + \ldots$$

at small r, where $a_{ij}^{(k)}$ is a constant. To avoid distracting algebra we shall take $a_{ij}^{(1)} = 0$ and $a_{ij}^{(2)} = 0$, so that a_{ij} may be regarded as constant. This does not affect our main result. It will modify only the recurrence relation for those coefficients in the series expansion of the wavefunctions which are higher than needed. First, we consider the case $a_{12}^{(0)} > 0$, which corresponds to the case of Kukulin *et al.*

Thus, we need to solve the equations

$$u'' + a_{11}u + a_{12}w = 0 \tag{1}$$

$$w'' + a_{12}u + a_{22}w - 6r^{-2}w = 0 \tag{2}$$

with a_{ij} taken as constant. Double differentiation of equation (1) and substitution of w and w'' leads to the following equation for u:

$$(r^2u'''' - 6u'') + [(a_{11} + a_{22})r^2u'' - 6a_{11}u] - r^2(a_{12}^2 - a_{11}a_{22})u = 0. \tag{3}$$

Alternatively, we can obtain the equation for w:

$$(r^4w'''' - 6r^2w'' + 24rw' - 36w) + r^2[(a_{11} + a_{22})r^2w'' - 6a_{11}w] - r^4(a_{12}^2 - a_{11}a_{22})w = 0. \tag{4}$$

Using the Frobenius method (see, e.g., Protter and Morrey 1969) for the solution of these equations, we write

$$u = \sum_{n=0}^{\infty} b_n r^{n+c}, \qquad b_0 \neq 0.$$

Equating to zero the coefficient of the lowest power of r in equation (3), we obtain the indicial equation, which has solutions

$$c = 0,\ 0,\ 1 \text{ or } 5. \tag{5}$$

The first two of these values give solutions which do not satisfy the boundary condition $u(r) \to 0$ as $r \to 0$ and can be discarded. We are left with the case in which the two values of the index differ by an integer, which gives rise to the possibility of a logarithmic term in r (Protter and Morrey 1969).

The same method for equation (4) and

$$w = \sum_{n=0}^{\infty} \bar{b}_n r^{n+\bar{c}}, \qquad \bar{b}_0 \neq 0,$$

leads to

$$\bar{c} = -2,\ 2,\ 3 \text{ and } 3. \tag{6}$$

The negative value can be discarded. One can show that $\bar{c} = 2$ corresponds to $c = 0$, so that this too can be discarded. The case of equal roots is more awkward than that when the roots differ by an integer, so we shall continue with the results (5) and show that (6) follows from them.

For $c = 5$, we have one solution of the form

$$u_1 = \sum_{n=0}^{\infty} b_n r^{n+5}, \qquad b_0 = 1. \tag{7}$$

The second solution has the form

$$u_2 = \alpha \ln r\, u_1 + \sum_{n=0}^{\infty} d_n r^{n+1}, \qquad d_0 = 1, \tag{8}$$

where α may be zero (Protter and Morrey 1969, Cornille 1962). Substituting the wavefunction (8) into equation (3), we find that

$$d_i = 0, \qquad \text{if } i \text{ is odd} \tag{9}$$

$$d_2 = -a_{11}/6 \tag{10}$$

and

$$\alpha = a_{12}^2/100. \tag{11}$$

Other coefficients may be obtained from lengthy recurrence relations which we do not give here. It follows that the general form for the wavefunction $u(r)$ is

$$u = A\left(0.01 a_{12}^2 \ln r \sum_{n=0}^{\infty} b_n r^{n+5} + \sum_{n=0}^{3} d_n r^{n+1}\right) + B \sum_{n=0}^{\infty} \bar{b}_n r^{n+5}, \tag{12}$$

with A and B as arbitrary constants and $\bar{b}_0 = b_0 = d_0 = 1$, where $\bar{b}_n$ contains b_n and d_{n+4}.

Substituting for u into equation (1), we obtain

$$\begin{aligned} w = -(A/a_{12})\Bigg(&0.01 a_{12}^2 \ln r \sum_{n=0}^{\infty} [(n+5)(n+4) + a_{11} r^2] b_n r^{n+3} \\ &+ 0.01 a_{12}^2 \sum_{n=0}^{\infty} (2n+9) b_n r^{n+3} + \sum_{n=0}^{3} [(n+1)n + a_{11} r^2] d_n r^{n-1}\Bigg) \\ &- (B/a_{12}) \sum_{n=0}^{\infty} [(n+5)(n+4) + a_{11} r^2] b_n r^{n+3}. \end{aligned} \tag{13}$$

The dominant terms for small r give, using equations (9) and (10)

$$w \simeq -(A/a_{12}) r^3 [0.01 a_{12}^2 (20 \ln r + 9) - a_{11}^2/6] - (B/a_{12}) 20 r^3. \tag{14}$$

We note that the terms r^3 and $r^3 \ln r$ reflect the double root of 3 for $\bar{c}$ (6).

If $A \neq 0$ we see that for small r,

$$u \simeq Ar \tag{15a}$$

$$w \simeq -0.2 A a_{12} r^3 \ln r \tag{15b}$$

so that

$$w/u \simeq -0.2 a_{12} r^2 \ln r,$$

which is *positive* for small r, since a_{12} is positive in the model of Kukulin *et al* (1984). This shows that w is negative because u is negative.

If $A = 0$, the second regular solution becomes important and then

$$u \simeq Br^5$$

$$w \simeq -(20/a_{12}) Br^3.$$

In general, a linear combination of the two results is necessary. It is only in a *very* special case that the boundary conditions will lead to $A = 0$. A similar result can be found in the simpler case when the term $(-6/r^2)$ is removed from equation (2). The exact solutions given by Kermode (1967) to a related problem have

$$u \simeq Cr, \qquad w \simeq Dr,$$

except in the special case $A_+k_+ + A_-k_- = 0$ (in the notation of that paper with $a = 0$), when

$$u \simeq Cr^3, \qquad w \simeq Dr.$$

We have considered the case $a_{12}^{(0)} > 0$. In the parametrisation of their potential (the Paris potential), Lacombe *et al* (1980) have, for small r, the case $a_{12}^{(0)} = 0$ and $a_{12}^{(1)} < 0$. (They have also $a_{11} \to a_{11}/r$ and $a_{22} \to a_{22}/r$, but this does not affect the present argument.) Following the previous procedure, we find $c = (0)$, 1, 1 and 6 and $\bar{c} = (-2)$, 3, 3 and 4. However, the simplest approach is to write $\bar{B} = (-20B + a_{11}^2 A/6)/a_{12}$ in equations (12) and (14) and then take the limit $a_{12} \to 0$ (with A and B remaining finite). This gives

$$u \simeq Ar \tag{16a}$$

$$w \simeq \bar{B}r^3 \tag{16b}$$

in agreement with the parametrisation of the Paris deuteron *wavefunctions* (Lacombe *et al* 1981). The term in $r^3 \ln r$ no longer appears since its second derivative contains $5r$ which would have no other term to cancel. Previously it was cancelled by the lowest power from the term $a_{12}u$. We note that A and $\bar{B}$ are *independent*. This is a special case.

For the Reid potential, a_{12} is replaced by $r^{-1}a_{12}$ and the wavefunctions behave like

$$u \simeq Ar \tag{17a}$$

$$w \simeq (A/4)a_{12}r^2 \tag{17b}$$

so again w/u is positive at small r (since $a_{12} > 0$).

3. Conclusion

We have considered the short-range behaviour of the deuteron wavefunctions arising from various forms for the tensor potential. In particular, we have shown that w/u is positive in the model of Kukulin *et al* because the tensor potential has a negative value at the origin. Thus, the node in the D-state wavefunction w is a consequence of both the node in the S-state wavefunction u and the fact that w/u is positive at large r. Kukulin *et al* indicated that it was necessary to have a non-local potential to obtain a negative value for w/u at small r. However, we see from equation (15*b*) that a negative w/u can be obtained if the tensor potential has a positive value at the origin. We know of no reason why $V_T(r)$ should not be repulsive at short distances, where the nucleon–nucleon interaction is very complicated.

In table 1, we summarise our results. These show the interesting behaviour of the ratio of the wavefunctions at short distances as $V_T(0)$ varies from $-\infty$ through to $+\infty$. It would be most interesting to find a model in which $V_T(0)$ is positive and the static properties of the deuteron are reproduced.

Table 1. The behaviour of w/u at short distances resulting from various values of the tensor potential at the origin. The fixed constant, C, is dependent on the value of $V_T(0)$ or $\lim_{r\to 0} rV_T(r)$.

$V_T(0)$	w/u	Example	Equation
$-\lim_{r\to 0}(r^{-1})$	$\lvert C\rvert r$	Reid (1968)	(17)
$-\infty < V_T(0) < 0$	$-\lvert C\rvert r^2 \ln r$	Kukulin *et al* (1984)	(15)
0	$\bar{B}r^3/Ar$	Paris (1980)	(16)
$0 < V_T < \infty$	$\lvert C\rvert r^2 \ln r$	none	(15)
$\lim_{r\to 0}(r^{-1})$	$-\lvert C\rvert r$	none	(17)

Acknowledgment

We are grateful to the SERC of the United Kingdom for support under grant NG 12938.

References

Cornille H 1962 *Nuovo Cimento* **23** 526-36
Friar J L and Fallieros S 1984 *Phys. Rev.* C **29** 232-9
Kermode M W 1967 *Nucl. Phys.* A **99** 605-24
Kukulin V I, Pomerantsev V N, Krasnopol'sky V M and Sazonov P B 1984 *Phys. Lett.* **135B** 20-4
Lacombe M, Loiseau B, Richard J M, Vinh Mau R, Côté J, Pirès P and deTourreil R 1980 *Phys. Rev.* C **21** 861-73
Lacombe M, Loiseau B, Vinh Mau R, Côté J, Pirès P and deTourreil R 1981 *Phys. Lett.* **101B** 139-40
Protter M H and Morrey C B 1969 *Modern Mathematical Analysis* (Reading, MA: Addison-Wesley) p 720
Reid R V 1968 *Ann. Phys., NY* **50** 411-48